

# Asymptotic Functions as Kernels of the Schwartz Distributions

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Using a version of the sequential method we introduce a class of generalized functions called here "asymptotic functions". This class contains kernels of all Schwartz distributions and is equipped with a correctly defined multiplication operation. So, in a sense, one solves the problem of "multiplication of Schwartz distributions" although the solution refers to the class of the asymptotic functions and not to the Schwartz distributions themselves. The paper is a continuation of a series of works [1—10] but here only part of the results of [5], [6] and [8] will be needed.

## Introduction

The purpose of the present work is to construct a class of generalized functions (different from the Schwartz distributions), called here "asymptotic functions", which have the following properties: (i) This class is a ring of functions and, in particular, every two asymptotic functions can be multiplied; (ii) Every Schwartz distribution has kernels in this class so, the expressions  $\delta^2(x)$ ,  $\delta(x) \cdot \Theta(x)$ ,  $\delta(x) \cdot X^{-n}$ ,  $\Theta(x)X^{-n}$ , etc. have sense in it where  $\delta(x)$ ,  $\Theta(x)$ ,  $X^{-n}$ , etc., are kernels of the corresponding Schwartz distributions. Recall that the products of the above mentioned type do not have sense in the conventional theory of distributions and, on the other hand, they appear in quantum mechanics and quantum field theory and cause some disadvantages in their mathematical ground [13]; (iii) The integrals (and the functionals as well) of the asymptotic functions turn out to be not real or complex numbers but "asymptotic numbers". The systems of the asymptotic numbers (real and complex, respectively) introduced by Christov [1], are systems of generalized numbers containing the ordinary numbers (real and complex, respectively), as well as infinitely small (infinitesimals) and infinitely large numbers [5, 6].

The reader could remain disappointed at the fact that a given Schwartz distribution may have more than one kernel in the class of the asymptotic functions  $F_0$ ; in fact, it has infinitely many kernels in  $F_0$ . For example, there are many asymptotic functions  $\delta \in F_0$  which are kernels of the Dirac's distribution  $\delta \in \mathcal{D}$  (we shall often use the same or nearly the same notations for a given Schwartz distribution and their kernels). But the lack of a one-to-one correspondence between  $F_0$  and  $\mathcal{D}$  is the "price" which must be "paid", according to us, if we want to achieve by all means multiplication operation.

The paper is a continuation of a series of works [1—10] and some knowledge at least of [5, 6] and [8] is presupposed. The class of asymptotic fun-

ctions  $F_0$  introduced here, contains all quasi-extended asymptotic functions  $F$ , defined in [10], i. e.  $F \subset F_0$  is valid. The reason for introducing this wider class of asymptotic functions lies in the fact that  $F_0$  contains kernels of all Schwartz distributions unlike the class of quasi-extended asymptotic functions  $F$  containing only some of them. Moreover, the construction of  $F_0$  is much more simple than the one of  $F$  and the definition of  $F_0$  is independent of  $F$ ; with the exception of Section 3 the paper could be understood without referring to [9] and [10].

The use of infinitely small and infinitely large numbers, respectively, the infinitesimal relation, is the reason why some terms and notions in the present paper are very much like those in the non-standard analysis [14] but the knowledge of the latter is not necessary for understanding our paper.

## 1. The Class of Regularization Functions V

A class of functions of two real variables will be introduced and the functions themselves will be treated as families of functions of one of the variables; the other one will play the role of a parameter of the families. An essential feature of this class is that every Schwartz distribution has a regularisation in this class and besides it is a ring, i. e. it is closed with respect to addition and multiplication. Then we are going to define an equivalence relation in this class and the equivalence classes will define "the asymptotic functions".

(1.1) Definition (The Class V):  $V$  will be the class of all functions of the type

$$(1.2) \quad v: \mathcal{R} \times (0,1) \rightarrow C$$

having the following properties:

(i) For every closed finite interval  $I$  of  $\mathcal{R}$  there exist an integer number  $m_I$  and a positive real number  $M_I$  ( $m_I$  and  $M_I$  depend on  $I$ ) such that

$$(1.3) \quad |v(x, \varepsilon)| \leq \frac{M_I}{\varepsilon^{m_I}}, \quad x \in I, \quad \varepsilon \in (0, 1);$$

(ii)  $v'_x$  belongs to  $V$  where  $v'_x$  is the derivative of  $v$  with respect to the first variable, i. e.

$$(1.4) \quad v'_x(x, \varepsilon) = \frac{\partial}{\partial x} v(x, \varepsilon), \quad x \in \mathcal{R} \quad \varepsilon \in (0, 1).$$

(1.5) Notations: By  $N$ ,  $Z$ ,  $\mathcal{R}$  and  $C$  we shall denote the sets of the natural, integer, real and complex numbers, respectively.

(1.6) Remark ( $V$  is a ring): Obviously,  $V$  is a ring with differentiation, i. e.  $V$  is closed with respect to addition and multiplication and  $x$ -differentiation.

(1.7) Example:  $V$  contains all functions of the type

$$(1.8) \quad v(x, \varepsilon) = \varepsilon^k e^{ipx} \psi\left(\frac{x-x_0}{\varepsilon^m}\right), \quad x \in \mathcal{R}, \quad \varepsilon \in (0, 1),$$

where  $\psi$  is a bound function from  $C^\infty$  ( $C^\infty$  is the class complex valued smooth functions defined on  $\mathcal{R}$ ,  $x_0, p \in \mathcal{R}$ , and  $k, m \in Z$ ).

(1.9) Example:  $V$  contains all functions of the type

$$(1.10) \quad v(x, \varepsilon) = \psi(x), \quad x \in \mathcal{R}, \quad \varepsilon \in (0, 1),$$

where  $\psi \in C^\infty$  ((1.10) follows from (1.8) for  $k=m=p=x_0=0$ ). In other words

$$(1.11) \quad C^\infty \subset V.$$

(1.12) Example (the jump-functions):  $V$  contains all functions of the type

$$(1.13) \quad v(x, \varepsilon) = w(x + i\varepsilon) - w(x - i\varepsilon), \quad x \in \mathcal{R}, \quad \varepsilon \in (0, 1),$$

where  $w$  belongs to the Tillmann's class of analytic functions [12].

(1.14) Remark (the Tillmann's class): Recall that the Tillmann's class  $H$  [12] contains (by definition) all analytic functions of the type

$$(1.15) \quad w: C \setminus \mathcal{R} \rightarrow C$$

having the properties: for every closed finite interval  $I$  of  $\mathcal{D}$  there exist an integer  $m_I$  and a positive constant  $M_I$  such that

$$(1.16) \quad |w(x + iy)| \leq \frac{M_I}{|y|^{m_I}}, \quad x \in I, \quad |y| \in (0, 1).$$

Recall, moreover, that all jump-functions of the type (1.13) (where  $w \in H$ ) are regularisations of some Schwartz distribution and on the contrary, any distribution has regularisation of the type (1.13) for some  $w \in H$ .

(1.17) Theorem ( $V$  and  $\mathcal{D}$ ):

(i) Every Schwartz distribution  $T \in \mathcal{D}'$  has a regularisation in  $V$ , i. e. there exists  $v \in V$  such that

$$(1.18) \quad \langle T, \varphi \rangle = \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} v(x, \varepsilon) \varphi(x) dx, \quad \varphi \in \mathcal{D};$$

(ii) If the limit

$$\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} v(x, \varepsilon) \varphi(x) dx$$

exists for some  $v \in V$  and all  $\varphi \in \mathcal{D}$ , then the functional  $T$ , defined by (1.18) is a Schwartz distribution, i. e.  $T \in \mathcal{D}'$ .

Proof: (i) Take  $v$  from example (1.12) and use the Tillmann's theorem (1.14);

(ii) (1.18) is obviously a linear functional on  $\mathcal{D}$ . We ought to show that  $T$  is a continuous one. The proof of the latter is quite similar to the proof of the Tillmann's theorem (12, Satz (3.3), p. 122) and we shall omit it.

(1.19) Definition (standard regularisation):

(i) The function  $v \in V$  will be called "standard" if it is of the type:

$$(1.20) \quad v(x, \varepsilon) = \varepsilon^{-m} \rho\left(\frac{x-x_0}{\varepsilon}\right), \quad x \in \mathcal{R}, \quad \varepsilon \in (0, 1),$$

for some  $m \in \mathbb{Z}$ , some  $x_0 \in \mathcal{R}$  and some bounded function  $\rho \in C^\infty$ ;

(ii) The distribution  $T \in \mathcal{D}'$  will be called "standard" if it possesses a standard regularisation in  $V$ , i. e. if (1.18) is valid for some standard  $v \in V$ .

(1.21) Remark: Many of the distributions (but not all) are standard. For example, the distributions  $\delta(x)$ ,  $\Theta(x)$ ,  $\delta_\pm(x)$  and  $P(x^{-n})$  and their derivatives turn out to be standard distributions.

(1.22) Definition (equivalence relation in  $V$ ):

(i) Two functions  $v, w \in V$  will be called equivalent if

$$(1.23) \quad \varepsilon^{-k} [v_x^{(m)}(x, \varepsilon) - w_x^{(m)}(x, \varepsilon)] \xrightarrow[\varepsilon \rightarrow 0]{I} 0$$

for every  $k, m \in \mathbb{Z}$ ,  $m \geq 0$ , and every finite closed interval  $I$  of  $\mathcal{R}$ , where " $\xrightarrow[\varepsilon \rightarrow 0]{I}$ " means "uniform limit on  $I$ " and  $v_x^{(m)}$  is the  $m$ -th derivative of  $v$  with respect to  $x$ , i. e.

$$(1.24) \quad v_x^{(m)}(x, \varepsilon) \equiv \frac{\partial^m}{\partial x^m} v(x, \varepsilon), \quad x \in \mathcal{R}, \quad \varepsilon \in (0, 1);$$

(ii) The set of all functions from  $V$  which are equivalent to the zero function will be denoted by  $V_0$ .

(1.25) Lemma (some properties of  $V_0$ ):

- (i)  $V_0$  is an ideal in  $V$ ;
- (ii)  $v \in V_0$  implies  $v'_x \in V_0$
- (iii)  $v \in V_0$  implies

$$(1.26) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{-k} \int_I v(x, \varepsilon) dx = 0$$

for any  $k \in \mathbb{Z}$  and any finite closed interval  $I$  of  $\mathcal{R}$ ;

(iv)  $v \in V_0$  implies  $v \cdot \varphi \in V_0$  for every boundary  $\varphi \in C^\infty$ .

Proof: Elementary.

(1.27) Corollary (the factor-space  $V/V_0$ ): The factor-space  $V/V_0$  (i.e. the space of all equivalence classes) is a ring with differentiation (just as  $V$ ).

## 2. Asymptotic Functions

(2.1) Definition (asymptotic functions): The factor-space  $V/V_0$  Section 1 will be denoted by  $F_0$ , i. e.

$$(2.2) \quad F_0 = V/V_0.$$

The elements of  $F_0$  (i. e. the equivalence classes) will be called "asymptotic functions" or simply " $F_0$ -functions" and for them the notations  $f, g, h, \dots$  or  $f(x), g(x), h(x)$ , etc. will be used. If  $v \in f \in F_0$ , then we shall write  $f = [v]$ .

(2.3) Definition (algebraic operations, differentiation, integration,  $\dots$ , etc. in  $F_0$ ): Let  $f, g \in F_0$ ,  $c \in C$ , and  $\sigma$  be a Lebesgue's measurable subset of  $\mathcal{R}$ . Then

$$(2.4) \quad f \pm g = [v \pm w],$$

$$(2.5) \quad cf = [cv],$$

$$(2.6) \quad f \cdot g = [v \cdot w],$$

$$(2.7) \quad f' = [v'_x]$$

$$(2.8) \quad \int_{\sigma} f(x) dx = \gamma_{as}(s),$$

where

$$(2.9) \quad \gamma(\varepsilon) = \int_{\sigma} v(x, \varepsilon) dx, \quad \varepsilon \in (0, 1),$$

and  $\gamma_{as}$  is the asymptotic extension of  $\gamma$  ([8], Section 3), and  $s$  is the asymptotic number defined by the formula

$$(2.10) \quad s = \{\varepsilon - \Delta(\varepsilon) \mid \lim_{\varepsilon \rightarrow 0} \varepsilon^{-k} \Delta(\varepsilon) = 0 \text{ for all } k \in \mathbb{Z}\}.$$

The Fourier-transforms and convolution are defined as follows:

$$(2.11) \quad \mathcal{F}(f) = [\mathcal{F}(v)], \quad \mathcal{F}^{-1}(f) = [\mathcal{F}^{-1}(v)]$$

and

$$(2.12) \quad f * g = [v * w],$$

respectively, where  $v \in f$  and  $w \in g$  are chosen arbitrarily,  $\mathcal{F}(v)$  and  $\mathcal{F}^{-1}(v)$  are the conventional Fourier-transforms of  $v$  and  $v * w$  is the conventional convolution of  $v$  and  $w$  (everywhere the second variable " $\varepsilon$ " of  $v$  and  $w$  is treated as an additional parameter). And finally, the composition  $f \circ g$  will be defined as follows:

$$(2.13) \quad f \circ g = [\varphi \circ \psi],$$

where

$$(2.14) \quad (\varphi \circ \psi)(x, \varepsilon) = \varphi(\psi(x, \varepsilon), \varepsilon), \quad x \in \mathcal{R}, \quad x \in (0, 1).$$

(The end of Definition (2.3)).

(2.15) Remark: Recall ([8], Section 2 and Section 3) that the asymptotic extension

$$\gamma_{as}: (0, 1)_{as} \rightarrow A^*$$

of a given continuous function of the type

$$\gamma: (0, 1) \rightarrow C,$$

is defined by the formula

$$(2.16) \quad \gamma_{as}(a) = as \{\gamma(\alpha) \mid \alpha \in a\}, \quad a \in (0, 1)_{as},$$

where "as" means "asymptotic cover" ([5], definition 7) and  $(0, 1)_{as}$  is the asymptotic extension of the real interval  $(0, 1)$  ([8], (2.18)). Recall moreover that  $A$  and  $A^*$  are the sets of the real and complex asymptotic numbers, respectively [5]. On the other hand, the asymptotic number  $s$  (2.10) used in the definition of integral (2.8) is a positive infinitesimal (6, Section 4), i. e.

$$(2.17) \quad 0 < s < \varepsilon$$

for any positive real number  $\varepsilon$ . Therefore  $s$  belongs to  $(0, 1)_{as}$  so  $\gamma_{as}(s)$  is a well-defined complex asymptotic number, i. e.

$$(2.18) \quad \int_{\sigma} f(x) dx \in A^*$$

in the cases the integral is convergent.

(2.19) Theorem (some properties of  $F_0$ ):

(i)  $F_0$  is a ring with differentiation, i. e.  $f, g \in F_0$ , and  $c \in C$  implies  $f + g, f \cdot g, cf, f' \in F_0$ ;

(ii) The integral (2.8) is convergent for all functions  $f$  from  $F_0$  if (atleast)  $\sigma = I$  is a finite closed interval of  $\mathcal{R}$ , i. e.

$$(2.20) \quad \int_I f(x) dx \in A^*$$

for all  $f \in F_0$  and all  $I$  of the above mentioned type.

Proof: See corollary (1.22).

(2.21) Remark: The integral (2.8), the Fourier-transforms (2.11), the convolution (2.12) as well as the composition (2.13) do not exist for all sets  $\sigma$  and all asymptotic functions from  $F_0$ .

(2.22) Lemma: If  $f = [v]$  for some  $v \in C^\infty$ , then the integral (2.8) coincides with the usual Lebesgue's integral of  $v$ , i. e.

$$(2.23) \quad \int_\sigma f(x) dx = \int_\sigma v(x) dx$$

(see example (1.9)).

Proof: Elementary.

(2.24) Corollary ( $C^\infty \subset F_0$ ): The mapping

$$(2.25) \quad C^\infty \ni v \rightarrow [v] = f \in F_0$$

is an isomorphism with respect to all operations defined in definition (2.3), i. e.

$$(2.26) \quad C^\infty \subset F_0$$

is an isomorphical embedment of  $C^\infty$  into  $F_0$ .

(2.27) Remark (interpretation): Lemma (2.22) and corollary (2.24) show us that the asymptotic functions of  $F_0$  could be treated as generalized functions of some type (which does not coincide with the Schwartz distributions).

(2.28) Definition (standard asymptotic functions): The asymptotic function  $f \in F_0$  will be called "standard" if  $v \in f$  for some standard function  $v \in V$  (1.19).

The advantage of the standard asymptotic functions (over the rest) is that the calculations of their derivatives, integrals and functionals are much more easily performed.

The following lemma will help us to calculate some integrals of the type (2.8) in some particular cases.

(2.29) Lemma: Let  $f \in F_0$  and  $\sigma$  be a Lebesgue measurable subset of  $\mathcal{R}$ . If the integral of  $v$  over  $\sigma$  for some  $v \in f$  has an asymptotic expansion

$$(2.30) \quad \int_\sigma v(x, \varepsilon) dx \underset{\varepsilon \rightarrow 0}{\sim} \sum_{k=\mu}^{\infty} c_k \varepsilon^k,$$

where  $\mu \in \mathbb{Z}$ ,  $c_k \in C$ ,  $k = \mu, \mu+1, \dots$ , then  $f$  is integrable on  $\sigma$  and

$$(2.31) \quad \int_\sigma f(x) dx = \sum_{k=\mu}^{\infty} c_k s^k,$$

where the convergence of the series in (2.31) is in the interval topology of  $A^*$  [6].

(2.32) Remark: Notice that the series (2.30) is not surely convergent (in the topology of  $C$ ) in contrast to the series (2.31) which is always (for any  $\mu$  and any  $c_k$ ) convergent in the topology of  $A^*$  ([6], Theorem 41).

Proof of the Lemma: (2.30) means (by definition) that  $\gamma$  (which is defined in (2.9)) could be represented in the form

$$\gamma(\varepsilon) = \sum_{k=\mu}^n c_k \varepsilon^k + \Delta_n(\varepsilon), \quad \varepsilon \in (0, 1),$$

for any  $n \in \mathbb{Z}$ ,  $n \geq \mu$ , and some  $\Delta_n$  such that

$$\lim_{\varepsilon \rightarrow 0} \Delta_n(\varepsilon)/\varepsilon^n = 0.$$

To come to (2.31) it is sufficient to use ([8], Lemma (3.12), (ii)) (where  $\varphi \equiv \gamma$ ,  $\bar{x} \equiv 0$ ,  $t \equiv \varepsilon$  and  $h \equiv s$ ). The proof is completed.

(2.33) Corollary: Let  $f \in F_0$  and let  $\sigma$  be a Lebesgue measurable subset of  $\mathcal{R}$ . If (2.30) is valid for some  $v \in f$ , then

$$(2.34) \quad \int_{\sigma} f(x) dx \approx \sum_{k=\mu}^0 c_k s^k,$$

where we have put  $\Sigma \equiv 0$  in the case  $\mu > 0$ .

(2.35) Remark: Recall ([6], Section 4) that if  $a, b \in A^*$  (i. e.  $a$  and  $b$  are two complex asymptotic numbers), then  $a \approx b$  if  $|a - b|$  is infinitesimal. The relation  $\approx$  is called "infinitesimal relation".

(2.36) Lemma: Let  $f \in F_0$  be a standard asymptotic (2.28) and let  $v \in f$ , where

$$(2.37) \quad v(x, \varepsilon) = \varepsilon^{-m} \rho\left(\frac{x - x_0}{\varepsilon}\right)$$

or some  $m \in \mathbb{Z}$ , some  $x_0 \in \mathcal{R}$  and some  $\rho \in \mathcal{S}$  ( $\mathcal{S}$  is the Schwartz space of test-functions). Let, finally,  $\varphi \in C^\infty$  be arbitrarily chosen. Then the formula

$$(2.38) \quad \int_{-\infty}^{\infty} f(x) \varphi(x) dx = \sum_{k=0}^{\infty} (-1)^k \varphi^{(k)}(x_0) M_k,$$

is valid (see again (2.32)), where

$$(2.39) \quad M_k = \frac{(-1)^k}{k!} \int_{-\infty}^{\infty} f(x) (x - x_0)^k dx, \quad k = 0, 1, \dots$$

Indeed, we obtain easily

$$\int_{-\infty}^{\infty} v(x, \varepsilon) \varphi(x) dx \sim \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(x_0)}{k!} \varepsilon^{k-m+1} \int_{-\infty}^{\infty} \rho(x) x^k dx$$

which, corresponding to Lemma (2.29), implies

$$\int_{-\infty}^{\infty} f(x) \varphi(x) dx = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(x_0)}{k!} s^{k-m+1} \int_{-\infty}^{\infty} \rho(x) x^k dx.$$

Analogously, the expansion

$$\int_{-\infty}^{\infty} v(x, \varepsilon) (x - x_0)^k dx \sim \varepsilon^{k-m+1} \int_{-\infty}^{\infty} \rho(x) x^k dx$$

implies the equality

$$\int_{-\infty}^{\infty} f(x)(x-x_0)^k dx = s^{k-1+1} \int_{-\infty}^{\infty} \rho(x)x^k dx.$$

Comparing the last formulae we obtain (2.38) and (2.39). Notice that

$$M_k = \frac{(-1)^k}{k!} s^{k-m+1} \int_{-\infty}^{\infty} \rho(x)x^k dx$$

are, in general, asymptotic numbers (but not surely usual real or complex numbers).

### 3. The Asymptotic Functions as Mappings

It is clear that the asymptotic functions of  $F_0$  are not pointwise functions in the set of the real and complex numbers. In other words, the functions from  $F_0$  are not mappings neither in the set  $\mathcal{R}$ , nor in  $C$ . However, we are able to introduce values of these functions in the set of the asymptotic numbers [5], i. e. to represent the functions from  $F_0$  by means of mappings in  $A$  or  $A^*$ .

The material exposed in the present section is not absolutely necessary to the understanding of the remaining part of this work. The reader who is not eager to know "what the values of the asymptotic functions are like" could omit reading this section.

(3.1) Lemma: If  $f \in F_0$  and  $v, w \in f$ , then

$$(3.2) \quad v_{as}(a, s) = w_{as}(a, s), \quad a \in \Omega,$$

where  $\Omega$  is the set of the finite real asymptotic numbers ([6], Section 4) and  $s$  is the infinitesimal number used in Section 2.

Proof:  $v, w \in f$  means that  $v$  and  $w$  are equivalent in  $V$  (1.19). Consequently, condition (1.20) holds which is equivalent (corresponding to a classical W. H. Young's theorem) to the following one:

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-k} [v_x^{(m)}(x + \Delta(\varepsilon), \varepsilon) - w_x^{(m)}(x + \Delta(\varepsilon), \varepsilon)] = 0$$

for all  $k, m \in \mathbb{Z}$ ,  $m \geq 0$ , all  $x \in \mathcal{R}$  and all mappings  $\Delta: (0, 1) \rightarrow \mathcal{R}$  such that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-n} \Delta(\varepsilon) = 0, \quad n \in \mathbb{Z}.$$

The above condition implies (3.2) corresponding to ([4], Teorem 30). The proof is completed.

(3.3) Definition (Graph): If  $f \in F_0$ , then the mapping

$$(3.4) \quad \tilde{f}: \Omega \rightarrow A^*$$

defined by the formula

$$(3.5) \quad \tilde{f}(a) = v_{as}(a, s), \quad a \in \Omega,$$

where  $v \in f$  is chosen arbitrarily, will be called "the graph of  $f$ ".

(3.6) Remark: The correctness of the above definition is ensured by Lemma (3.1).

(3.7) Remark: The graphs of the asymptotic functions of  $F_0$  are quasi-extended asymptotic functions corresponding to the terminology introduced in [10].



(3.8) Theorem: Let  $f, g \in F_0$  and let  $\tilde{f}$  and  $\tilde{g}$  be their graphs, respectively. Then  $\tilde{f} + \tilde{g}$  and  $\tilde{f} \cdot \tilde{g}$  are the graphs of  $f + g$  and  $f \cdot g$ , respectively.

(3.9) Remark: We are not going to give the proof of the above theorem (it is closely connected with ([10], Theorem (4.1))). We shall only stress that  $f + g$  and  $f \cdot g$  are defined in the framework of  $F_0$  and  $\tilde{f} + \tilde{g}$  and  $\tilde{f} \cdot \tilde{g}$  are defined point-wisely in the class of the quasi-extended asymptotic functions  $F$  [10], i. e.

$$(3.10) \quad (\tilde{f} + \tilde{g})(a) = \tilde{f}(a) + \tilde{g}(a), \quad a \in \Omega,$$

and

$$(3.11) \quad (\tilde{f} \cdot \tilde{g})(a) = \tilde{f}(a) \cdot \tilde{g}(a), \quad a \in \Omega.$$

(3.12) Remark: Let  $f$  be a standard asymptotic function (2.28) and let  $v \in f$ , where  $v$  is given by the formulae (1.20) (see the text of Definition (1.19)). Then the graph  $\tilde{f}$  of  $f$  is given by the formula

$$(3.13) \quad \tilde{f}(a) = s^{-m} \rho_{as} \left( \frac{a - x_0}{s} \right), \quad a \in \Omega,$$

where

$$\rho_{as}: \Omega \rightarrow A^*$$

is the asymptotic extension of  $\rho$  ([8], section 3). Notice that  $\tilde{f}$  determines uniquely  $\rho$  by the formulae

$$(3.14) \quad \rho(x) = s^m \tilde{f}(x_0 + sx), \quad x \in \mathcal{R}.$$

Bearing in mind (1.20), we conclude that there is a one-to-one correspondence between  $f$  and its graph  $\tilde{f}$ .

## 4. Asymptotic Distributions

In the present section we are going to establish a connection between the asymptotic functions from the class  $F_0$  and the Schwartz distributions.

(4.1) Definition (asymptotic functionals): Let  $\Phi$  be any of the spaces of test-functions  $\mathcal{E} \equiv C^\infty$ ,  $\mathcal{O}_\alpha$ ,  $\alpha \in \mathcal{R}$ ,  $\mathcal{S}$  or  $\mathcal{D}$  which are used in the Schwartz theory of distributions [11] (the spaces are provided with the corresponding test-topology). Then

(i) The mapping of the type

$$(4.2) \quad T: \Phi \rightarrow A^*$$

will be called "asymptotic functionals on  $\Phi$ ";

(ii) An asymptotic functional  $T$  will be called "linear" or "quasi-linear" if the conditions

$$(4.3) \quad T(c_1 \varphi_1 + c_2 \varphi_2) = c_1 T(\varphi_1) + c_2 T(\varphi_2)$$

or

$$(4.4) \quad T(c_1 \varphi_1 + c_2 \varphi_2) \approx c_1 T(\varphi_1) + c_2 T(\varphi_2)$$

are valid respectively for all  $c_1, c_2 \in C$  and all  $\varphi_1, \varphi_2 \in \Phi$ , where " $\approx$ " is the infinitesimal relation in the system of the asymptotic numbers ([6], Section 4);

(iii)  $T$  will be called "continuous" if

$$(4.5) \quad \lim_{n \rightarrow \infty} T(\varphi_n) = T(\varphi)$$

for all  $\varphi_n, \varphi \in \Phi, n \in N$ , for which

$$(4.6) \quad \varphi_n \xrightarrow[n \rightarrow \infty]{\Phi} \varphi,$$

where the limit " $\xrightarrow[n \rightarrow \infty]{\Phi}$ " is in the sense of the test-topology of  $\Phi$  and "Lim" is the limit of the interval topology of  $A^*$  ([6], Section 5);

(iv) The derivative  $T'$  of a given asymptotic functional  $T$  will be defined by the formula

$$(4.7) \quad T'(\varphi) = -T(-\varphi'), \quad \varphi \in \Phi.$$

(4.8) Remark: Recall that  $A$  and  $A^*$  are the sets of the real and complex asymptotic numbers, respectively ([5], Section 1, Section 2). Recall moreover that the inclusions

$$\mathcal{R} \subset A, \quad C \subset A^*, \quad A \subset A^*$$

are valid and consequently, the Schwartz distributions are a particular type of asymptotic functionals.

(4.9) Definition (asymptotic distributions):

(i) An asymptotic functional  $T$  defined on  $\Phi$  (4.1) will be called "asymptotic distribution on  $\Phi$ " if there exists an asymptotic function  $f \in F_0$  such that

$$(4.10) \quad T(\varphi) \approx \int_{-\infty}^{\infty} f(x)\varphi(x)dx, \quad \varphi \in \Phi.$$

The asymptotic function  $f$  will be called "kernel" of  $T$  and the set of all kernels of  $T$  will be denoted by  $\mathcal{K}_T$ ;

(ii) The set of all asymptotic distributions on  $\Phi$  will be denoted by  $\Phi^0$  (e. g.  $\mathcal{E}_0, \mathcal{O}_a^0, S^0, \mathcal{D}^0$  etc);

(iii) If (4.10) is valid for some standard asymptotic function  $f$  (2.28), then  $T$  will be called "standard asymptotic distribution".

(4.11) Example ( $F_0$ -functionals): The asymptotic functionals of the type

$$(4.12) \quad T_f(\varphi) = \int_{-\infty}^{\infty} f(x)\varphi(x)dx, \quad \varphi \in \Phi,$$

where  $f \in F_0$ , give examples of asymptotic distributions on  $\Phi$ . We shall say that " $T_f$  is determined by  $f$ ".  $T_f$  will be called " $F_0$ -functional".

(4.13) Example (Schwartz distributions): We shall show further that the Schwartz distributions are a particular type of asymptotic distributions from the space  $\mathcal{D}^0$  (4.9). Notice that the Schwartz distributions are not (in general) of the type (4.12) (excepting the cases of regular distributions, of course).

(4.14) Theorem (some properties):

(i) Every asymptotic distribution  $T$  is a quasi-linear asymptotic functional;

(ii) If  $f, f_1, f_2 \in F_0$  are kernels of  $T, T_1, T_2 \in \Phi^0$ , respectively, then  $f_1 + f_2$  and  $cf$  are kernels of  $T_1 + T_2$  and  $cT$ , respectively;

(iii) If  $f \in F_0$  is a kernel of  $T \in \Phi^0$ , then  $f^{(m)}$  is a kernel of  $T^{(m)}$ ,  $m=0, 1, \dots$ , i. e. the formula

$$(4.15) \quad T^{(m)}(\varphi) \approx \int_{-\infty}^{\infty} f^{(m)}(x)\varphi(x)dx, \quad \varphi \in \Phi$$

is valid;

(iv) The inclusions

$$(4.16) \quad \mathcal{E}^0 \subset \mathcal{O}_\alpha^0 \subset \mathcal{O}_\beta^0 \subset S^0 \subset \mathcal{D}^0$$

are valid for  $\alpha > \beta$ ,  $\alpha, \beta \in \mathcal{R}$ .

Proof: (i) (4.10) leads to

$$T(c_1 \varphi_1) \subset c_1 \int_{-\infty}^{\infty} f(x) \varphi_1(x) dx,$$

$$T(c_2 \varphi_2) \approx c_2 \int_{-\infty}^{\infty} f(x) \varphi_2(x) dx,$$

$$T(c_1 \varphi_1 + c_2 \varphi_2) \approx \int_{-\infty}^{\infty} f(x) [c_1 \varphi_1(x) + c_2 \varphi_2(x)] dx$$

from which (4.4) follows immediately;

(ii) The inclusions (4.15) are derived analogously to the corresponding inclusions of  $\mathcal{E}'$ ,  $\mathcal{O}'_\alpha$ ,  $S'$  and  $\mathcal{D}'$  in the Schwartz theory of distributions [11];

(iii) follows directly from (4.11);

(iv) We shall derive (4.15) only for  $m=1$ . Let (4.10) hold for some  $T \in \Phi^0$  and some  $f \in K_T$ . That means, corresponding to ([6], Theorem 30, (i)), that

$$(4.17) \quad \lim_{\varepsilon \rightarrow 0} \left\{ t_\varphi(\varepsilon) - \int_{-\infty}^{\infty} v(x, \varepsilon) \varphi(x) dx \right\} = 0,$$

$$\varphi \in \Phi, \quad \varepsilon \in (0, 1),$$

for any  $v \in f$  and any  $t_\varphi \in T(\varphi)$  ( $t_\varphi$  is a function of the type  $t_\varphi: (0, 1) \rightarrow C$ ; recall that every asymptotic number is, by definition, a set of such kind of functions [5]). Changing " $\varphi$ " by " $-\varphi$ " in (4.17), we obtain

$$(4.18) \quad \lim_{\varepsilon \rightarrow \infty} \left\{ t_{-\varphi}(\varepsilon) - \int_{-\infty}^{\infty} v'_x(x, \varepsilon) \varphi(x) dx \right\} = 0, \quad \varphi \in \Phi,$$

where  $t_{-\varphi} \in T(-\varphi)$  is arbitrarily chosen. But (3.18) is equivalent to

$$T'(\varphi) \approx \int_{-\infty}^{\infty} f'(x) \varphi(x) dx, \quad \varphi \in \Phi,$$

having in mind (4.7). The formula (4.15) implies (4.15) by induction. The proof is completed.

(4.19) Remark: The asymptotic distributions are not (in general) continuous functionals in contrast to the Schwartz distributions.

Bearing in mind the inclusions (4.15), we shall restrict ourselves to the case  $\Phi = \mathcal{D}$  only ( $\mathcal{D}$  is the space of the smooth complex-valued test-functions with compact supports defined on  $\mathcal{R}$ ).

(4.20) Theorem ( $F_0$  and  $\mathcal{D}$ ):

(i) Every Schwartz distribution has a kernel in  $F_0$ ;

(ii) A given asymptotic function  $f \in F_0$  cannot be a kernel of more than one Schwartz distribution.

Proof: (i) We have already mentioned (1.17) that any distribution has a regularisation in the class  $V$  (1.1). So, let  $T$  be a Schwartz distribution, i. e.  $T \in \mathcal{D}'$ , and let  $v \in V$  be its regularisation, i. e.

$$(4.21) \quad \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} v(x, \varepsilon) \varphi(x) dx = \langle T, \varphi \rangle, \quad \varphi \in \mathcal{D}.$$

The latter is equivalent to (4.10) for  $f = [\varphi]$  and  $T(\varphi) = \langle T, \varphi \rangle$ ;

(ii) Let  $f$  be a kernel of  $T_1, T_2 \in \mathcal{D}'$ . We have

$$\langle T_1, \varphi \rangle \approx \langle T_2, \varphi \rangle, \quad \varphi \in \mathcal{D},$$

which implies

$$\langle T_1, \varphi \rangle = \langle T_2, \varphi \rangle, \quad \varphi \in \mathcal{D},$$

since  $\langle T_1, \varphi \rangle$  and  $\langle T_2, \varphi \rangle$  are complex numbers (0 is the only infinitesimal among the complex numbers). In other words,  $T_1 = T_2$ . The proof is completed

(4.22) Corollary: The Schwartz distributions are asymptotic distributions (4.9), i. e.

$$(4.23) \quad \mathcal{D}' \subset \mathcal{D}^0.$$

(4.24) Remark: A given Schwartz distribution has (in general) infinitely many kernels in  $F_0$ .

(4.25) Remark (interpretation): The kernels  $f \in F_0$  of a given distribution  $T \in \mathcal{D}'$  will be treated further as an analog, or as a representation of  $T$  in  $F_0$ .

(4.26) Theorem: Let  $f \in F_0$  and let the values of the functional  $T_f$  on  $\mathcal{D}$  (4.12) be finite asymptotic numbers ([6], Section 4), i. e.

$$(4.27) \quad |T_f(\varphi)| = \left| \int_{-\infty}^{\infty} f(x) \varphi(x) dx \right| \in \Omega, \quad \varphi \in \mathcal{D},$$

where  $\Omega$  is the set of the finite (but not infinitely large) asymptotic numbers. Then  $f$  determines a unique Schwartz distribution  $T$  by the formula

$$(4.28) \quad \langle T, \varphi \rangle \approx \int_{-\infty}^{\infty} f(x) \varphi(x) dx, \quad \varphi \in \mathcal{D}.$$

(4.29) Remark: Recall ([6], Section 4), that any finite asymptotic number  $\omega \in \Omega$  is infinitely close to a unique real (or complex) number, i. e. there exists a unique  $r \in \mathcal{R}$  (or  $r \in \mathcal{C}$ ) such that  $\omega \approx r$ . So, (4.28) really determines a unique complex-valued functional defined on  $\mathcal{D}$ .

Proof: The above theorem is, in fact, a corollary of Theorem (1.17), (ii). Indeed, (4.27) and (4.28) are equivalent to the existence of the limit

$$\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} v(x, \varepsilon) \varphi(x) dx$$

for some  $v \in f$  and all  $\varphi \in \mathcal{D}$ , which, on its hand, implies  $T \in \mathcal{D}'$ , corresponding to the above-mentioned theorem. The proof is completed.

(4.30) Remark ( $F_0$  instead of  $\mathcal{L}^0$ ): The asymptotic distributions cannot be multiplied just like the Schwartz distributions (we mean, of course, such kind of multiplication which generalized the usual multiplication of the smooth functions of  $C^\infty$ ). But the kernels of the asymptotic distributions (including the kernels of the Schwartz distributions) can be multiplied correctly in the class  $F_0$  since, as shown in (2.19),  $F_0$  is a ring of functions. That is the reason why we shall abandon the space of the asymptotic distributions  $\mathcal{D}^0$  and the space of the Schwartz distributions  $\mathcal{D}'$  as well, and replace them with the class of asymptotic functions  $F_0$ . So that we shall treat the class  $F_0$  as "a substitute" of  $\mathcal{D}^0$ , respectively,  $\mathcal{D}'$ , and pay attention first of all to the properties (and to some applications they might have) of the asymptotic functions of  $F_0$ .

(4.31) Remark (which functions from  $F_0$  are interesting?): In connection with the previous remark the following question arises: Which asymptotic functions of the class  $F_0$  will be considered to be "interesting ones" and, respectively, which asymptotic functions will be in the centre of our attention further on?

(i) First of all we shall be interested, of course, in these asymptotic functions from  $F_0$  which are kernels of some Schwartz distributions because of the important role of the latter in mathematics and its applications;

(ii) Second, the products of any kernels of the Schwartz distributions (which are asymptotic functions from  $F_0$  as well) will be especially studied for these products, on the one hand, do not have any analogs among the Schwartz distributions (excepting the cases of the regular distributions, of course) and, on the other hand, these products give sense of the formulas:  $\delta^2(x)$ ,  $\delta(x) \cdot x^{-n}$ ,  $\delta(x)\Theta(x)$ ,  $\Theta(x) \cdot X^{-n}$ , etc. having significance in quantum field theory;

(iii) Finally, we shall prefer to work with the standard asymptotic functions from  $F_0$  (2.28) just because the integrals and functionals of such a kind of functions can be more easily calculated.

(4.32) Remark (the role of the  $F_0$ -functionals): The  $F_0$ -functionals, i. e. the asymptotic functionals of the type

$$T_f(\varphi) = \int_{-\infty}^{\infty} f(x)\varphi(x) dx, \quad \varphi \in \mathcal{D},$$

where  $f \in F_0$ , will be used as an indirect way to describe the properties of the function  $f$ . We shall often choose  $f$  in the above functional to be an asymptotic function of the type mentioned in (i), (ii) and (iii) of Remark (4.31).

(4.33) Remark (the generalization to the asymptotic functions of  $n$ -variables): All definitions and results exposed so far can be easily generalized to the  $n$ -dimensional case, i. e. to the case of asymptotic functions of  $n$ -variables. Some difficulties in the construction of the jump-functions of  $n$ -variables (which are analogous to those from example (1.12)) arise but we shall refer the reader to them in ([11], Chapter V).

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